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Convergence of Generalized Bernstein Polynomials

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Let $f \in C[0, 1]$, $q \in (0, 1)$, and $B_n(f, q; x)$ be generalized Bernstein polynomials based on the q-integers. These polynomials were introduced by G. M. Phillips in 1997. We study convergence properties of the sequence $\{B_n(f, q; x)\}_{n=1}^{\infty}$. It is shown that in general these properties are essentially different from those in the classical case $q = 1$.

Key Words: generalized Bernstein polynomials; q-integers; q-binomial coefficients; convergence.

1. INTRODUCTION

In 1912 Bernstein [2] found the proof of the Weierstrass Approximation Theorem based on the Law of Large Numbers for a sequence of Bernoulli trials. He constructed, for any continuous function $f \in C[0, 1]$, a sequence of polynomials

$$B_n(f; x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}, \qquad n = 1, 2, \ldots \tag{1}$$

and proved that the sequence converges to f for $n \to \infty$ uniformly with respect to $x \in [0, 1]$. These polynomials (1), called *Bernstein polynomials*, possess many remarkable properties. They have been studied intensively,

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and their connections with different branches of analysis, such as convex and numerical analysis, total positivity, and the theory of monotone operators, have been investigated. Due to the fact that $\{B_n(f; x)\}$ is an approximating sequence of shape-preserving operators, Bernstein polynomials play an important role in computer-aided geometric design [4]. We mention also recent applications of Bernstein polynomials in the theory of multidimensional probability distributions [5]. Basic facts on Bernstein polynomials, their generalizations and applications, can be found in, e.g., [6, 11].

In 1997 Phillips [8] introduced generalized Bernstein polynomials $B_n(f, q; x)$ based on the q-integers and q-binomial coefficients for any $q > 0$. For $q = 1$, q-binomial coefficients and generalized Bernstein polynomials coincide with the classical ones. For $q \neq 1$, one gets a new class of polynomials having interesting properties. Generalized Bernstein polynomials attracted much interest and were studied widely by Goodman *et al.* in [3, 7–10]. They obtained a great number of results devoted to various properties of these polynomials.

In this paper we study problems of convergence for generalized Bernstein polynomials. Phillips [8] was the first person to investigate these problems. In particular, he obtained analogs of Bernstein's and Voronovskaya's results for generalized Bernstein polynomials (5).

In this paper we obtain new results related to convergence properties of these polynomials (5). Our results demonstrate that in general these properties are essentially different from those in the classical case (cf. Theorems 3, 4, and 5). Our approach differs from Phillips'; like Bernstein's, it is based on some probabilistic considerations.

2. STATEMENT OF RESULTS

We need the following definitions.

Let $q > 0$. For any $n = 0, 1, 2, \ldots$ the *q-integer* $[n]_q$ is defined as

$$[n]_q := 1 + q + \cdots + q^{n-1} \qquad (n = 1, 2, \ldots), \quad [0]_q := 0; \tag{2}$$

and the *q-factorial* $[n]_q!$ as

$$[n]_q! := [1]_q\, [2]_q \ldots [n]_q \qquad (n = 1, 2, \ldots), \quad [0]_q! := 1. \tag{3}$$

For integers $0 \leqslant k \leqslant n$ the *q-binomial* or the *Gaussian coefficient* is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q!\, [n-k]_q!}. \tag{4}$$

DEFINITION (Phillips [8]). Let $f \in C[0,1]$. *The generalized Bernstein polynomial based on the q-integers* is

$$B_n(f,q;x) := \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k} (1-q^s x), \qquad n=1,2,\dots. \tag{5}$$

(From here on an empty product is taken to be equal 1.)

Note that for $q=1$ we obtain the classical Bernstein polynomials (1).

Let $B_n(f,q;x)$ be defined by (5). It is shown in [8] that

$$B_n(at+b,q;x) = ax+b \qquad \text{for all} \quad q>0 \quad \text{and all} \quad n=1,2,\dots. \tag{6}$$

It follows directly from (5) that

$$B_n(f,q;0) = f(0); \qquad B_n(f,q;1) = f(1) \qquad \text{for all} \quad q>0$$
$$\text{and all} \quad n=1,2,\dots \tag{7}$$

The following analog of Bernstein's Theorem for polynomials (5) is due to G. M. Phillips.

THEOREM A. *Let a sequence q_n satisfy $0<q_n<1$ and $q_n \to 1$ as $n \to \infty$. Then for any function $f \in C[0,1]$,*

$$B_n(f,q_n;x) \rightrightarrows f(x) \qquad [x \in [0,1]; n \to \infty].$$

The expression $g_n(x) \rightrightarrows g(x)$ $[x \in [0,1]; n \to \infty]$ denotes convergence of g_n to g uniformly with respect to $x \in [0,1]$.

In this paper we also present a new proof of Theorem A (cf. Theorems 1 and 2).

In the sequel we always assume that $q \in (0,1)$ and f is a real continuous function on [0,1].

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, and $Z\colon \Omega \to \mathbf{R}$ be a random variable. We use the standard notation $\mathsf{E}Z$ for the mathematical expectation and Var Z for the variance of the random variable Z:

$$\mathsf{E}Z := \int_\Omega Z(\omega)\, \mathsf{P}(d\omega); \qquad \mathsf{Var}\, Z := \mathsf{E}(Z^2) - (\mathsf{E}Z)^2.$$

We set

$$p_{nk}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k} (1-q^s x), \qquad n=1,2,\dots. \tag{8}$$

Obviously $p_{nk}(q; x) \geqslant 0$ for $q \in (0, 1)$ and $x \in [0, 1]$. It follows from (6) (for $a = 0$, $b = 1$) that

$$\sum_{k=0}^{n} p_{nk}(q; x) = 1 \qquad \text{for all} \quad n = 1, 2, ... \tag{9}$$

Consider a random variable $Y_n(q; x)$ having the probability distribution

$$\mathsf{P}\left\{ Y_n(q; x) = \frac{[k]_q}{[n]_q} \right\} = p_{nk}(q; x), \qquad k = 0, 1, ...n; \quad n = 1, 2, ... \tag{10}$$

Evidently, $B_n(f, q; x) = \mathsf{E}f(Y_n(q; x))$.

It is not difficult to see that the limits as $n \to \infty$ of both the values of $Y_n(q; x)$ and the probabilities of these values exist. Indeed, for all $k = 0, 1, ...$

$$\lim_{n \to \infty} \frac{[k]_q}{[n]_q} = 1 - q^k, \tag{11}$$

$$\lim_{n \to \infty} p_{nk}(q; x) = \frac{x^k}{(1-q)^k [k]_q!} \prod_{s=0}^{\infty} (1 - q^s x) =: p_{\infty k}(q; x). \tag{12}$$

Note that unlike $p_{nk}(q; x)$ the functions $p_{\infty k}(q; x)$ are transcendental entire functions rather than polynomials. Obviously $p_{\infty k}(q; x) \geqslant 0$ for $x \in [0, 1]$ and by Euler's Identity (cf. [1, Chap. 2, Corollary 2.2]) we have

$$\sum_{k=0}^{\infty} p_{\infty k}(q; x) = 1 \qquad \text{for all} \quad x \in [0, 1). \tag{13}$$

Therefore, we can consider the random variables $Y_\infty(q; x)$ given by the following probability distributions:

$$\begin{aligned} &\mathsf{P}\{Y_\infty(q; x) = 1 - q^k\} = p_{\infty k}(x), \qquad k = 0, 1, ... \text{ for } x \in [0, 1), \\ &\mathsf{P}\{Y_\infty(q; 1) = 1\} \qquad = 1. \end{aligned} \tag{14}$$

(The latter distribution arises naturally since $Y_\infty(q; x) \to Y_\infty(q; 1)$ in probability as $x \uparrow 1$.)

For $f \in C[0, 1]$ we set

$$B_\infty(f, q; x) := \mathsf{E}f(Y_\infty(q; x)).$$

It follows from (14) that

$$B_\infty(f, q; x) = \begin{cases} \sum_{k=0}^{\infty} f(1-q^k)\, p_{\infty k}(q; x), & \text{if } x \in [0, 1), \\ f(1), & \text{if } x = 1. \end{cases} \tag{15}$$

Our main results on convergence are Theorems 1 and 2 below.

THEOREM 1. *For any* $f \in C[0, 1]$,

$$B_\infty(f, q; x) \rightrightarrows f(x) \qquad [x \in [0, 1];\ q \uparrow 1].$$

THEOREM 2. *Let* $0 < \alpha < 1$. *Then for any* $f \in C[0, 1]$

$$B_n(f, q; x) \rightrightarrows B_\infty(f, q; x) \qquad [x \in [0, 1],\ q \in [\alpha, 1];\ n \to \infty].$$

That is, $B_n(f, q; x)$ converges to $B_\infty(f, q; x)$ for $n \to \infty$ uniformly with respect to $x \in [0, 1]$ and $q \in [\alpha, 1]$. Evidently Theorem A follows from Theorems 1 and 2.

It is natural to ask how close properties of f and $B_\infty(f, q; x)$ are. The following result due to Phillips [8] is an immediate corollary of Theorem 1.

COROLLARY 1. *If f is a polynomial of degree $\leqslant m$, then $B_\infty(f, q; x)$ is also a polynomial of degree $\leqslant m$.*

We show that a stronger assertion holds:

THEOREM 3. *If f is a polynomial, then* $\deg B_\infty(f, q; x) = \deg f$.

So, the function $B_\infty(f, q; x)$ is the limit of the sequence of generalized Bernstein polynomials $B_n(f, q; x)$ when $q \in (0, 1)$ is fixed. The following theorem treats the smoothness of $B_\infty(f, q; x)$.

We say that $f \in C[0, 1]$ satisfies *the Lipschitz condition at the point* 1 if there exist $\alpha > 0$, $M > 0$ such that

$$|f(t) - f(1)| \leqslant M\, |t-1|^\alpha \qquad \text{for} \quad t \in [0, 1].$$

THEOREM 4. *For any $f \in C[0, 1]$ the function $B_\infty(f, q; x)$ is continuous on $[0, 1]$ and analytic in the unit disk $\{x: |x| < 1\}$. If f satisfies the Lipschitz condition at* 1, *then $B_\infty(f, q; x)$ is differentiable from the left at* 1.

Generally $B_\infty(f, q; x)$ may not be differentiable at 1. An illustrative example is given after the proof of Theorem 3.

Using the explicit form (15) of the function $B_\infty(f, q; x)$, we derive the following unicity theorem.

THEOREM 5. *If $f(1-q^k)=0$ for all $k=0, 1, 2, \ldots$, then $B_\infty(f, q; x)=0$ on $[0, 1]$. If $B_\infty(f, q; x)=0$ for an infinite number of points having an accumulation point on $[0, 1)$, then $f(1-q^k)=0$ for all $k=0, 1, 2\ldots.$*

COROLLARY 2. *$B_\infty(f, q; x)=0$ for $x \in [0, 1]$ if and only if $f(1-q^k)=0$ for all $k=0, 1, 2\ldots.$*

COROLLARY 3. *Let f be a polynomial. Then $B_\infty(f, q; x)=0$ if and only if $f=0$.*

Let $q \in (0, 1)$ be fixed. The following theorem describes completely the class of continuous functions satisfying the condition

$$\lim_{n \to \infty} B_n(f, q; x) = f(x) \qquad \text{for} \quad x \in [0, 1].$$

THEOREM 6. *Let $f \in C[0, 1]$. Then $B_\infty(f, q; x)=f(x)$ for all $x \in [0, 1]$ if and only if $f(x)=ax+b$, where a and b are constants.*

It can be readily seen from (15) that for a fixed $q \in (0, 1)$ there exist different continuous functions $f \neq g$ such that $B_\infty(f, q; x)=B_\infty(g, q; x)$. This is because $B_\infty(f, q; x)$ is defined only by the values of f at the points $\{1-q^k\}_{k=0}^{\infty}$. In particular, there exist non-linear functions f such that $B_\infty(f, q; x)$ are linear functions.

However, the following statement holds.

THEOREM 7. *Let $f \in C[0, 1]$ and*

$$B_\infty(f, q_j; x) = a_j x + b_j \qquad (x \in [0, 1])$$

for a sequence q_j such that $q_j \uparrow 1$. Then f is a linear function.

3. PROOF OF THE THEOREMS

Proof of Theorem 1. Since by (7) and (15),

$$B_n(f, q; 1) = B_\infty(f, q; 1) = f(1)$$

for all $q > 0$, it suffices to prove that

$$B_\infty(f, q; x) \rightrightarrows f(x) \qquad [x \in [0, 1), q \uparrow 1].$$

Let us denote

$$\psi := \psi(q; x) := \prod_{s=0}^{\infty} (1-q^s x) \tag{16}$$

and write for $k = 1, 2, \ldots$

$$p_{\infty k}(q; x) = \frac{x^k \psi}{(1-q)(1-q^2)\ldots(1-q^k)}.$$

By direct calculations we get

$$\begin{aligned}\mathsf{E}(Y_\infty(q; x)) &= \sum_{k=1}^{\infty} (1-q^k)\, p_{\infty k}(q; x)\\ &= x\left(\psi + \sum_{k=2}^{\infty} \frac{x^{k-1}\psi}{(1-q)\ldots(1-q^{k-1})}\right) = x \sum_{j=0}^{\infty} p_{\infty j}(q; x) = x,\end{aligned}$$

and

$$\begin{aligned}\mathsf{E}((Y_\infty(q; x))^2) &= \sum_{k=1}^{\infty} (1-q^k)^2\, p_{\infty k}(q; x) = \sum_{k=1}^{\infty} \frac{(1-q+q-q^k)\, x^k \psi}{(1-q)\ldots(1-q^{k-1})}\\ &= (1-q)\, x \sum_{k=0}^{\infty} p_{\infty k}(q; x) + q x^2 \sum_{k=0}^{\infty} p_{\infty k}(q; x)\\ &= (1-q)\, x + q x^2.\end{aligned}$$

Hence

$$\mathsf{Var}(Y_\infty(q; x)) = (1-q)\, x(1-x) \leqslant (1-q)/4$$

and it tends to 0 uniformly with respect to $x \in [0, 1]$ as

$q \uparrow 1$. Now we show that

$$B_\infty(f, q; x) = \mathsf{E} f(Y_\infty(q; x)) \rightrightarrows f(x) \qquad [x \in [0, 1), q \uparrow 1].$$

Let $\varepsilon > 0$ be given. We choose $\delta > 0$ in such a way that $|f(t') - f(t'')| < \varepsilon/2$ for $|t' - t''| < \delta$, $t', t'' \in [0, 1]$. We denote $C = \max\{|f(t)|: t \in [0, 1]\}$ and $A = \{\omega \in \Omega : |Y_\infty(q; x) - x| \geqslant \delta\}$. Since by Chebyshev's Inequality

$$\mathsf{P}(|Y_\infty(q; x) - x| \geqslant \delta\}) \leqslant \delta^{-2}\, \mathsf{Var}\, Y_\infty(q; x),$$

we obtain

$$|B_\infty(f, q; x) - f(x)| \leqslant \left(\int_A + \int_{\Omega \setminus A}\right) |f(Y_\infty(q; x) - f(x)| \, \mathsf{P}(d\omega)$$
$$\leqslant 2C\mathsf{P}(|Y_\infty(q; x) - x| \geqslant \delta\}) + \varepsilon/2$$
$$\leqslant 2C\delta^{-2} \operatorname{Var} Y_\infty(q; x) + \varepsilon/2$$
$$\leqslant \delta^{-2}(1-q)/2 + \varepsilon/2 < \varepsilon,$$

if q is sufficiently close to 1. ∎

Proof of Theorem 2. First, we prove the following lemmas.

LEMMA 1. *For any $\varepsilon > 0$ there exists a small $\eta_\varepsilon > 0$ and a positive integer N_ε such that*

$$|B_n(f, q; x) - f(x)| < \varepsilon$$

for all $x \in [0, 1]$, $q \in [1-\eta_\varepsilon, 1)$ and $n > N_\varepsilon$.

Remark. This is G. M. Phillips' Theorem A.

Proof of Lemma 1. It is proved in [8] that

$$B_n(t, q; x) = x, \qquad B_n(t^2, q; x) = x^2 + x(1-x)/[n]_q.$$

This means that

$$\mathsf{E}Y_n(q; x) = x, \qquad \mathsf{E}Y_n(q; x)^2 = x^2 + x(1-x)/[n]_q.$$

Hence

$$\operatorname{Var} Y_n(q; x) = x(1-x)(1+q+q^2+\cdots+q^{n-1})^{-1}.$$

Let $|f(x)| \leqslant C$ for all $x \in [0, 1]$. Let $\delta > 0$ be chosen in such a way that $|f(t) - f(x)| \leqslant \varepsilon/2$, whenever $|t - x| \leqslant \delta$. Applying the Chebyshev Inequality, we get

$$|B_n(f, q; x) - f(x)| \leqslant \int_{[0,1]} |f(t) - f(x)| \, P_{Y_n(q;x)}(dt) \leqslant \int_{|t-x| \leqslant \Delta} + \int_{|t-x| > \Delta}$$
$$\leqslant \varepsilon/2 + 2C\delta^{-2} \operatorname{Var} Y_n(q; x)$$
$$\leqslant \varepsilon/2 + C\delta^{-2}(1+q+q^2+\cdots+q^{n-1})^{-1}.$$

We set $\eta_\varepsilon = \varepsilon\delta^2/2C$ and we take N_ε in such a way that for all $n \geqslant N_\varepsilon$ the following inequality holds

$$1+(1-\eta_\varepsilon)+(1-\eta_\varepsilon)^2+\cdots+(1-\eta_\varepsilon)^{n-1} \geqslant 1/2(1-(1-\eta_\varepsilon)) = 1/2\eta_\varepsilon.$$

Then for $q \geqslant 1-\eta_\varepsilon$, $n \geqslant N_\varepsilon$ and all $x \in [0, 1]$ we have

$$|B_n(f, q; x) - f(x)| \leqslant \varepsilon/2 + C\delta^{-2}\eta_\varepsilon = \varepsilon.$$

Lemma 1 is proved.

LEMMA 2. *Let* $0 < \alpha < \beta < 1$, *and let* $p_{nk}(q; x)$ $(k = 0, 1, ..., n,\ n = 1, 2, ...)$ *and* $p_{\infty k}(q; x)$ $(k = 0, 1, ...)$ *be functions given by* (8) *and* (12), *respectively.*

Then for any $k = 0, 1, 2, ...$

$$p_{nk}(q; x) \rightrightarrows p_{\infty k}(q; x) \qquad [x \in [0, 1],\ q \in [\alpha, \beta];\ n \to \infty].$$

Proof of Lemma 2. We note that $\left[{n \atop k}\right]_q \to ((1-q)^k [k]_q!)^{-1}$ as $n \to \infty$ uniformly with respect to $q \in [\alpha, \beta]$. Therefore, it suffices to prove that

$$\prod_{s=0}^{n-1-k} (1-q^s x) \to \prod_{s=0}^{\infty} (1-q^s x) \qquad (n \to \infty)$$

uniformly with respect to $q \in [\alpha, \beta]$. This follows from the estimate:

$$0 \leqslant \prod_{s=0}^{n-1-k} (1-q^s x) - \prod_{s=0}^{\infty} (1-q^s x)$$

$$\leqslant 1 - \prod_{s=n-k}^{\infty} (1-q^s x) \leqslant 1 - \prod_{s=n-k}^{\infty} (1-\beta^s) \to 0, \qquad n \to \infty.$$

∎

Now, let $\varepsilon > 0$ be given. By Theorem 1 there exists a small number $\zeta_\varepsilon > 0$ such that for all $x \in [0, 1]$ and all $q \in [1-\zeta_\varepsilon, 1)$ we have

$$|B_\infty(f, q; x) - f(x)| \leqslant \varepsilon.$$

Let $\eta_\varepsilon > 0$ and N_ε be numbers pointed out in Lemma 1. We set $\xi = \min\{\eta_\varepsilon, \zeta_\varepsilon\}$. Then for all $x \in [0, 1]$, $n > N_\varepsilon$ and $q \in [1-\xi_\varepsilon, 1)$ we obtain

$$|B_n(f, q; x) - B_\infty(f, q; x)| \leqslant 2\varepsilon.$$

To complete the proof of the Theorem it suffices to show that $B_n(f, q; x) \to B_\infty(f, q; x)$ uniformly with respect to $x \in [0, 1)$ and $q \in [\alpha, 1-\xi_\varepsilon]$, because by (7) and (15),

$$B_n(f, q; 1) = f(1) = B_\infty(f, q; 1)$$

for all q.

We choose $a \in (0, 1)$ in such a way that $|f(t) - f(1)| < \varepsilon/3$ for $a \leqslant t \leqslant 1$. Let R be a positive integer satisfying the condition $1 - q^{R+1} \geqslant a$ for all $q \in [\alpha, 1-\xi_\varepsilon]$. We estimate the difference

$$\Delta := |B_n(f, q; x) - B_\infty(f, q; x)|$$

for $n > R$ and $x \in [0, 1)$. Using (9) and (13) we get

$$\begin{aligned}
\Delta &= \left| \sum_{k=0}^{n} \left(f\left(\frac{[k]_q}{[n]_q}\right) - f(1) \right) p_{nk}(q; x) - \sum_{k=0}^{\infty} \left(f(1-q^k) - f(1)\right) p_{\infty k}(q; x) \right| \\
&\leqslant \left| \sum_{k=0}^{R} \left(f\left(\frac{[k]_q}{[n]_q}\right) - f(1) \right) p_{nk}(q; x) - \sum_{k=0}^{R} \left(f(1-q^k) - f(1)\right) p_{\infty k}(q; x) \right| \\
&\quad + \sum_{k=R+1}^{n} \left| f\left(\frac{[k]_q}{[n]_q}\right) - f(1) \right| p_{nk}(q; x) + \sum_{k=R+1}^{\infty} |f(1-q^k) - f(1)|\, p_{\infty k}(q; x) \\
&=: S_1 + S_2 + S_3.
\end{aligned}$$

Using Lemma 2 and the fact $f([k]_q/[n]_q) \to f(1-q^k)$ as $n \to \infty$ for all $k = 1, 2, \ldots, R$ uniformly with respect to $q \in [\alpha, 1-\xi_\varepsilon]$, we conclude that $S_1 < \varepsilon/3$ for n sufficiently large. Using (9) and positivity of $p_{nk}(q; x)$ we get

$$S_2 < \frac{\varepsilon}{3} \sum_{k=R+1}^{n} p_{nk}(q; x) \leqslant \varepsilon/3.$$

Similarly

$$S_3 < \frac{\varepsilon}{3} \sum_{k=R+1}^{\infty} p_{\infty k}(q; x) \leqslant \frac{\varepsilon}{3}.$$

Thus, $\Delta < \varepsilon$. ∎

Proof of Theorem 3. We use induction on $m = \deg f$. For $m = 1$ the statement is true by (6). Let us suppose that the statement is true for $\deg f \leqslant m$ and consider $B_\infty(t^{m+1}, q; x)$. We get by (16)

$$\begin{aligned} B_\infty(t^{m+1}, q; x) &= \sum_{k=1}^{\infty} (1-q^k)^{m+1} \frac{x^k \psi}{(1-q)\dots(1-q^{k-1})(1-q^k)} \\ &= \sum_{k=1}^{\infty} ((1-q)+q(1-q^{k-1}))^m \frac{x^k \psi}{(1-q)\dots(1-q^{k-1})} \\ &= \sum_{k=0}^{m} \binom{m}{k} q^k (1-q)^{m-k}\, x \sum_{r=1}^{\infty} (1-q^{r-1})^k \frac{x^{r-1}\psi}{(1-q)\dots(1-q^{r-1})} \\ &= \sum_{k=0}^{m} \binom{m}{k} q^k (1-q)^{m-k}\, x B_\infty(t^k, q; x). \end{aligned}$$

By the induction assumption this is a polynomial of degree $m+1$. ∎

Proof of Theorem 4. Continuity of $B_\infty(f, q; x)$ with respect to x on $[0, 1]$ follows immediately from the fact that $B_\infty(f, q; x)$ is a limit of a uniformly convergent sequence of polynomials (cf. Theorem 1). To prove analyticity we write for $|x| < 1$,

$$B_\infty(f, q; x) = \psi(q; x) \sum_{k=0}^{\infty} \frac{f(1-q^k)}{(1-q)\dots(1-q^k)} x^k, \tag{17}$$

where $\psi(q; x)$ defined by (16) is an entire function. Note that for $k=0$ the denominator in (17) is taken to be 1. Since

$$\lim_{k\to\infty} (1-q)(1-q^2)\dots(1-q^k) = \prod_{s=1}^{\infty} (1-q^s) \neq 0,$$

it follows that the sequence $\{f(1-q^k)/\prod_{s=1}^{k} (1-q^s)\}_{k=1}^{\infty}$ is bounded. Thus the sum in (17) is an analytic function for $|x| < 1$ and so is $B_\infty(f, q; x)$. Now assume that f satisfies Lipschitz condition at the point 1. We prove that in this case $B_\infty(f, q; x)$ is differentiable from the left at 1. Using (15), (13), and (16), we get

$$\begin{aligned} B_\infty(f, q; x) - B_\infty(f, q; 1) &= B_\infty(f, q; x) - f(1) \\ &= \sum_{k=0}^{\infty} (f(1-q^k) - f(1))\, p_{\infty k}(q; x) \\ &= \psi(q; x) \sum_{k=0}^{\infty} \frac{(f(1-q^k) - f(1))}{(1-q)^k [k]_q!} x^k \\ &= (1-x)\, \psi_1(q; x)\, u(q; x), \end{aligned}$$

where

$$\psi_1(q; x) := -\prod_{s=1}^{\infty} (1-q^s x); \qquad u(q; x) := \sum_{k=0}^{\infty} \frac{(f(1-q^k)-f(1))}{(1-q)^k [k]_q!}\, x^k. \tag{18}$$

Since the sequence $\{((1-q)^k\,[k]_q!)^{-1}\}_{k=0}^{\infty}$ is bounded and $|f(1-q^k)-f(1)| \leqslant M(q^\alpha)^k$, it follows that the series in (18) is uniformly convergent on $[0, 1]$. Hence the function $u(q; x)$ is continuous on $[0, 1]$. Thus

$$\lim_{x \uparrow 1} \frac{B_\infty(f, q; x) - B_\infty(f, q; 1)}{x-1} = \psi_1(q; 1)\, u(q; 1)$$

and so $B_\infty(f, q; x)$ is differentiable at 1 from the left. ∎

EXAMPLE. Let $f \in C[0, 1]$ and $f(0) = f(1) = 0$, $f(1-q^k) = 1/k$, $k = 1, 2, \ldots$. Then by (18),

$$\lim_{x \uparrow 1} \psi_1(q; x) = \psi_1(q; 1) \neq 0 \qquad \text{and} \qquad \lim_{x \uparrow 1} u(q; x) = \infty.$$

Thus

$$\lim_{x \uparrow 1} \frac{B_\infty(f, q; x) - B_\infty(f, q; 1)}{x-1} = \infty$$

and $B_\infty(f, q; x)$ is not differentiable at 1.

Proof of Theorem 5. If $f(1-q^k) = 0$ for all $k = 0, 1, 2, \ldots$, then by (15), $B_\infty(f, q; x) \equiv 0$. Conversely, let $B_\infty(f, q; x) = 0$ for an infinite number of points x having an accumulation point on $[0, 1)$. Since by Theorem 3 the function $B_\infty(f, q; x)$ is analytic for $|x| < 1$, the statement follows from the Unicity Theorem for analytic functions. ∎

Proof of Theorem 6. If $f(x) = ax + b$, then by (6) $B_n(f, q; x) = ax + b = f(x)$ for all $n = 1, 2, \ldots$ and therefore

$$B_\infty(f, q; x) = \lim B_n(f, q; x) = f(x).$$

Now we suppose that $B_\infty(f, q; x) = f(x)$ for every $x \in [0, 1]$. Let us consider the function $g(x) := f(x) - (f(1) - f(0))\, x$. It is evident that $g(0) = g(1)$ and $B_\infty(g, q; x) = g(x)$. We will prove that $g(x) = g(0) = g(1)$ for all $x \in [0, 1]$. Let $M := \max_{x \in [0, 1]} g(x)$. Now assume that $M > g(1)$, then $M = g(z)$ for some $z \in (0, 1)$ and $g(1-q^k) < M$ for k sufficiently large. Using (13) and positivity of $p_{\infty k}(q; x)$ we have

$$M = g(z) = \sum_{k=0}^{\infty} g(1-q^k)\, p_{\infty k}(q; z) < M.$$

The contradiction shows that $g(x) \leqslant g(1)$ for all $x \in [0, 1]$ Similarly we prove that $g(x) \geqslant g(1)$ for all $x \in [0, 1]$. Thus $g(x) \equiv b$ for some $b \in \mathbf{R}$ and finally $f(x) = ax + b$. ∎

Proof of Theorem 7. Let $B_\infty(f, q_j; x) = a_j x + b_j$ for all $x \in [0, 1]$. From (6) and Theorem 5 it follows that $f(x) = a_j x + b_j$ for $x \in \{1 - q_j^k\}_{k=0}^{\infty}$. Since $q_j \uparrow 1$ and $f \in C[0, 1]$, we get $f(x) = ax + b$ for some $a, b \in \mathbf{R}$. ∎

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